

## **Nuclear Spectral Energy Functions Beyond the Shell Model**

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We predict  $l=0$  nucleons in  $^{12}\text{C}$  to have a negative (binding) energy centered around  $-22$  MeV with a full width at half-maximum of 5.3 MeV. The  $l=1$  ( $P_{3/2}$  nucleons) are predicted to have a much narrower spectral energy function centered around  $-10.6$  MeV. A strongly correlated translational invariant wave function was used to describe the ground state nucleus. A central two-nucleon potential was utilized in the hyperspherical harmonic method to approximately solve the Schrödinger equation for the ground state wave function. Both confirmation and failings of the independent particle shell model are exposed.

### **INTRODUCTION**

Previous calculations (Gross, 1970; Wille, 1971; Faessler, 1973; Koltun, 1974; and Fritsch, 1975) of the nucleon spectral energy function have been based on the independent particle shell model with residual interactions. A sharp peak in the calculated spectral energy function can be associated with each hole, one-particle-two-hole, or two-particle-three-hole configuration that can couple to the target wave function. In contrast tightly bound hole states experimentally have a large width (Mougey, 1976; Makamura, 1976), while only the low binding hole states are sharply peaked. High-energy nucleon knock-out reactions on nuclei, such as  $(e, ep)$ ,  $(p, 2p)$  show that for fixed scattering angles, the summed energy of the final state projectiles has a continuous distribution. The one-step direct interaction model of these reactions allows one to infer the spectral energy function from the summed final-state energy spectrum. With a well-defined incident projectile energy, the independent particle shell model predicts a sharp distribution of final state energies, and a spectral energy function of

$$S_p(W) = n_p \delta(W - E_p) \quad (1)$$

Here  $n_\nu$  is the number of nucleons with quantum numbers  $\nu$  in the target ground state, and  $E_\nu$  is the single-particle (negative) energy. The spectral energy function predicted by the independent particle shell model is a sum of discrete delta functions of energy. It is the purpose of this paper to calculate a nuclear spectral function that has a continuous distribution in energy. We drop the independent particle motion assumption of the shell model and use instead a strongly interacting translation-invariant hyper-spherical harmonic wave function.

We have calculated a spectral energy function for the  $l=0$  and  $l=1$  nucleons in the ground state of  $^{12}\text{C}$ . We have used a phenomenological two-nucleon potential with the nonrelativistic 12-body Schrödinger equation. The solution to the Schrödinger equation is expanded in the hyper-spherical harmonic expansion (Baz, 1970; Sadovoi, 1975; Strobel, 1976; Strobel, 1978) and only the configuration corresponding to a full  $S_{1/2}$  shell and a full  $P_{3/2}$  shell is retained. This approximation limits us to comparing widths of deep and loosely bound hole states, and prevents any attempt to reproduce the  $^{12}\text{C}$  spectral energy function in all its details. We separate the momentum and energy dependence of the spectral energy function as

$$S(\mathbf{P}, W) = \sum_{\nu} |\varphi_{\nu}(\mathbf{p})|^2 S_{\nu}(W) \quad (2)$$

Neglecting recoil energies,  $W$  is the energy supplied to the nucleus to remove a proton.  $\mathbf{P}$  is the momentum of the proton within the target nucleus. The spectral function  $S(\mathbf{p}, W)$  is the joint probability of finding a proton with momentum  $\mathbf{p}$  in the target and with energy  $W$ . The subscript  $\nu$  denotes the quantum numbers for a bound nucleon of  $(n, l, j, j_z, \tau z)$ . The summation of  $\nu$  in equation (1) is effectively terminated by summing over the occupied states of the assumed configuration of the target. In the approximation calculated, we include only  $s_{1/2}$  ( $l=0$ ) and  $p_{3/2}$  ( $l=1$ ) nucleons for the ground state configuration of  $^{12}\text{C}$ . This paper will present equations for calculating the factors of the spectral energy function appearing in equation (1), starting from a two-nucleon potential.

The Hamiltonian for the  $^{12}\text{C}$  nucleus ( $A=12$ ) is assumed to be

$$H = -\frac{\hbar^2}{2m} \Delta_{\xi} + \sum_{i < j} V(r_{ij}) \quad (3)$$

where the center-of-mass kinetic energy has been removed and  $\Delta_{\xi}$  is the Laplacian expressed in the Jacobi coordinates. It can be related to the usual Laplacian via

$$\Delta_{\xi} = \sum_{i=1}^A \nabla_{r_i}^2 - \nabla_R^2 / A$$

where

$$\mathbf{R} = \sum_{i=1}^A \mathbf{r}_i / A \quad (4)$$

The solution to

$$H\Psi = E\Psi \quad (5)$$

where  $E$  is the total binding energy of the  $^{12}\text{C}$  nucleus with respect to six protons and six neutrons is assumed known and adequately approximated by a  $K$  minimal hyperspherical calculation. Rather than using a Slater determinant product type independent particle wave function, we expand the nuclear wave function in hyperspherical harmonics. Because center-of-mass motion is properly handled, a simple product wave function is not obtained in this expansion. Each term in the expansion has a hyperradial dependence that does not separate.

Thus the wave function is expanded as

$$\Psi = \sum_{K\alpha} \chi_{K\alpha}(\rho) Y_{K\alpha}(\Omega) / \rho^{(3A-4)/2} \quad (6)$$

where the  $Y_{K\alpha}(\Omega)$  are antisymmetrical Slater determinants of homogenous hyperspherical harmonic polynomials, and the  $\chi_{K\alpha}$  are the corresponding hyperradial functions. Here the hyperradius is defined as

$$\rho^2 = \sum_{i=1}^A (\mathbf{r}_i - \mathbf{R})^2 / A \quad (7)$$

Substituting equation (6) into the 12-body Schrödinger equation with the center of mass removed, equation (5) and retaining only the terms corresponding to a filled  $S_{1/2}^4 P_{3/2}^8$  configuration, results in

$$\frac{-\hbar^2}{2m} \left[ \frac{d^2}{d\rho^2} - \frac{\mathcal{L}(\mathcal{L}+1)}{\rho^2} \right] \chi + [W(\rho) - E] \chi = 0 \quad (8)$$

where

$$W(\rho) = \int Y_{K\alpha}^*(\Omega) \sum_{i < j} V(r_{ij}) Y_{K\alpha}(\Omega) d\Omega \quad (9)$$

and

$$\mathcal{L} = K + (3/2)(A - 2) = 23 \quad (10)$$

Here the subscripts  $Ka$  are dropped since only one value for each is retained in equation (8).

We assume a potential with the spin-isospin dependence of the one-pion Yukawa potential, and a space part given by

$$V(r) = -A(e^{-x} + Be^{-3x} - Ce^{-6x})/x \quad (11)$$

where  $x = \mu_\pi r$ .  $C$  is adjusted to produce a repulsive potential  $r_{12}$  less than 0.5 fermi in the  $S=1, T=0$  two-nucleon state.  $A$  and  $B$  are adjusted to match the total binding energy of  $^{12}\text{C}$  to the eigenvalue of equation (5). Thus the potential is strictly phenomenological and central ( $A=6$  MeV,  $B=1.6$ ,  $C=4.9$ ). For closed  $j$  shell nuclei, the tensor and spin orbit parts average to zero so, to the extent that  $^{12}\text{C}$  is a closed  $j$  shell nucleus, one can use a central potential in seeking the total binding energy.

The spectral energy function is usually defined as

$$S_\nu(W) = \langle \Psi | a_\nu^* \delta(W - H) a_\nu | \Psi \rangle \quad (12)$$

where  $a_\nu^+, a_\nu$  are creation and annihilation operators, respectively, for a nucleon with the single-particle set of quantum numbers  $\nu$ . These creation and annihilation operators are associated with a product wave function (Fetter, 1971; Bohr, 1969; De Shalit, 1974) and the matching Hamiltonian is then a sum of one- and two-particle wave function matrix elements times the appropriate creation and annihilation operators. A product wave function of one-body wave functions is not used here; consequently, the standard creation annihilation operator formalism is not used either. We define the probability a nucleon in quantum state  $\nu$  has energy  $W$  as:

$$S_\nu(W) = \langle \Psi_a | \delta(W - H_\nu) | \Psi_a \rangle \quad (13)$$

where  $H_\nu$  includes only certain of the terms of the total Hamiltonian, equation (3). We require that the coordinates of the particle in quantum state  $\nu$  be operated on by the term in the total Hamiltonian, for that term to be included in  $H_\nu$ . Neglecting center-of-mass effects, this definition is equivalent to equation (12).

This requirement is straightforward for distinguishable particles. For instance, if the coordinates of the particle in quantum state  $\nu$  are labeled  $r_1$ , then

$$H_\nu = -\frac{\hbar^2}{2m} \nabla_{r_1}^2 + \sum_j V(r_{1j}) \quad (14)$$

When the particle removed is one of several identical particles, antisymmetry of the wave function upon identical particle exchange complicates the

operator  $H$ , somewhat. The principle that led to equation (14) remains. We just require the coordinates of the included terms in the Hamiltonian to match the coordinates of the particle in quantum state  $\nu$ . Thus we have  $A$  body matrix elements to evaluate in equation (13) for the spectral energy function.

### THEORY

Here we outline the details of the solution for the  $A$ -body wave function  $\Psi$ , and how to get the one-body wave function  $\phi_\nu(\mathbf{p})$  from it, as well as the spectral energy function  $S_\nu(W)$ . The  $A$ -body wave function is expanded in hyperspherical coordinates as

$$\Psi_a = \sum_{Ka} \chi_{Ka}(\rho) Y_{Ka}(\Omega) / \rho^{(3A-4)/2} \equiv \sum_{Ka} R_{Ka}(\rho) Y_{Ka}(\Omega) \quad (15)$$

where the  $Y_{Ka}(\Omega)$  are orthonormalized antisymmetric hyperharmonic polynomials and the  $\chi_{Ka}$  satisfy coupled differential equations:

$$\left\{ \left( -\frac{\hbar^2}{2m} \right) \left[ \frac{d^2}{d\rho^2} - \frac{\mathcal{L}(\mathcal{L}+1)}{\rho^2} \right] + W(\rho) - E \right\} \chi_{Ka}(\rho) = \sum_{K'a' \neq Ka} W_{K'a'}^{Ka}(\rho) \chi_{K'a'}(\rho) \quad (16)$$

$W_{K'a'}^{Ka}$  is the hyperspherical angular average of the potential:

$$W_{K'a'}^{Ka}(\rho) = \int d\Omega Y_{K'a'}^* \sum_{i < j} V(r_{ij}) Y_{Ka} \quad (17)$$

The hyperradius  $\rho$  satisfies

$$\rho^2 = \sum_{i < j} (\mathbf{r}_i - \mathbf{r}_j)^2 / A = \sum_{i=1}^A r_i^2 - AR^2 = \sum_{i=1}^{A-1} \xi_i^2 \quad (18)$$

The volume element can be written as

$$d\tau_{3A-3} = \prod_{i=1}^{A-1} d\xi_i = \rho^{3A-4} d\rho d\Omega \quad (19)$$

and the integral  $\int d\Omega$  can be transformed to

$$\int d\Omega = \frac{1}{\pi i} \int_C \frac{dS}{\rho^{3A-S}} \left(\frac{S}{\pi}\right)^{3/2} \exp\left[S\left(\rho^2 - \sum_{i=1}^A r_i^2\right)\right] d\tau_{3A} \quad (20)$$

This relation is quite useful because it permits treating the  $3A$  coordinates as independent, allowing most integrals to be performed analytically. Conservation of momentum is assured by doing the contour integral  $ds$  prior to the last integral to be evaluated. The contour  $C$  is up the imaginary  $S$  axis from  $-\infty$  to plus  $\infty$ , with an infinitesimal positive real part.

The antisymmetry of the wave function is guaranteed via the hyperspherical polynomials  $Y_{ka}(\Omega)$ . For minimal  $K$ , they are defined in general as

$$Y_{ka}(\Omega) = BP_{ka} / \rho^K (A!)^{1/2} \quad (21)$$

Here  $B$  is a normalization factor

$$B^2 = \pi^{3/2} \Gamma(x) / 2 \quad (22)$$

and

$$x = K + 3(A - 1) / 2 \quad (23)$$

The  $P_{ka}$  are Slater determinants where

$$K = \sum_{i=1}^A (2n + l)_i \quad (24)$$

and  $a$  denotes the set of single-particle quantum numbers  $\nu = (n, l, j, m, tz)$  assigned to each of the  $A$  nucleons, as well as the coupling scheme, if needed, for the configuration  $a$ . The monomials of the Slater determinant are

$$\Phi_\nu(\mathbf{r}) = C_{nl} r^{2n+l} \sum_{m_l m_s} Y_{lm_l}(\hat{r}) \chi_{ms}^{1/2} \{lm_l, \frac{1}{2}ms; jm\} \chi_{\tau_z}^{1/2} \quad (25)$$

where  $\{lm_l, \frac{1}{2}ms; jm\}$  is a Clebsch-Gordan coefficient coupling  $l$  and  $s$  to total angular momentum  $j$  and  $z$  component  $m$ . The  $\chi_{ms}, \chi_{tz}$  are nucleon

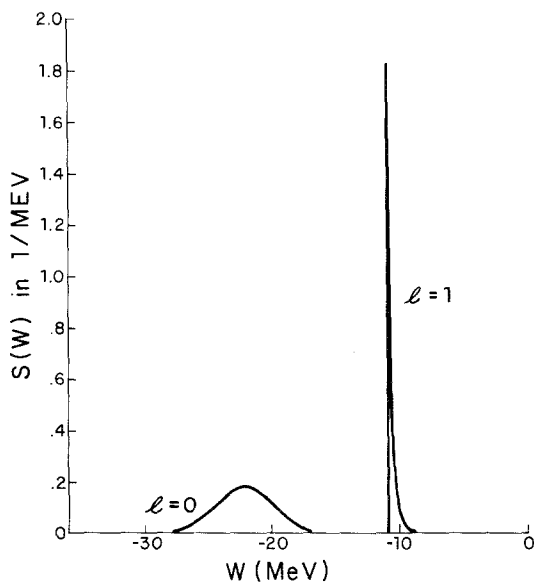


Fig. 1. The hyperradial dependence calculated for the ground state of  $^{12}\text{C}$  using the two-nucleon potential of equation (11) of the text.

spinors and isospinors. The  $C$  is a coefficient, chosen for convenience to be

$$C^2 = 2 / (2n + l + \frac{1}{2})! \quad (26)$$

For the minimal  $K$  configuration we consider ( $K_{\min} = 8$ ) of ( $S_{1/2}^4 P_{3/2}^8$ ), all the  $n$  values are zero and these will be dropped where possible in future equations. Because of bias towards an attractive nuclear spin orbit potential, configurations including any nucleons in  $p_{1/2}$  or "higher" orbitals  $\nu$  are neglected. Thus the sum of  $K'a'$  is neglected in equation (16) and an approximate  $\chi(\rho)$  and  $E$  obtained from its uncoupled residue equation (8). The two-nucleon potential is adjusted until the experimental binding energy is obtained using this configuration only. The solution for the hyperradial dependence  $\chi_{K_a}(\rho)$  is shown in Figure 1.

In this section the relation of the single-particle wave function  $\phi_r(\mathbf{r})$  to the many-body solution to the Schrödinger equation is outlined. We work in coordinate space and fourier transform to momentum space at the end. Here let  $\mathbf{R}$  denote a distance from the center of mass of the entire nucleus to the nucleon.

Now define the density distribution for this entire nucleus as

$$d(R) = \int d\hat{R} \int d\rho d\Omega \Psi_{K_a}^* \sum_{n=1}^A \delta^3(\mathbf{R} - \mathbf{r}_n) \Psi_{K_a} \quad (27)$$

This can be rewritten, using equation (20) for  $d\Omega$  as

$$d(R) = \frac{B^2}{A! \pi i} \iint \frac{d\hat{R} d\rho ds}{\rho^{3A-5+2K}} \prod_{l=1}^A d\mathbf{r}_l \left(\frac{S}{\pi}\right)^{3/2} \exp\left[S\left(\rho^2 - \sum_{i=1}^A r_i^2\right)\right] \chi_{Ka}^*(\rho) \\ \times \sum_{i=1}^A \phi_\nu^*(\mathbf{r}_i) (-1)^{i+\nu} C^{*\nu r_i} \sum_{n=1}^A \delta(\mathbf{R} - \mathbf{r}_n) \sum_{j=1}^A (-1)^{j+\nu} \phi_\nu(r_j) C^{\nu r_j} \chi_{Ka}(\rho) \quad (28)$$

Here we have used equation (20) for  $d\Omega$  and expanded the Slater determinants in cofactors, labeled  $C^{\nu r}$  where each cofactor is an  $(A-1)$  by  $(A-1)$  square determinant itself. The density distribution for a nucleon with quantum labels  $\nu$  is found by including only those terms in the sum over  $n$  where  $n=j$ . Thus

$$d_\nu(R) = \frac{B^2}{A! \pi i} \iint \frac{d\hat{R} d\rho ds}{\rho^{2K+3A-5}} \prod_{i=1}^A d\mathbf{r}_i \left(\frac{S}{\pi}\right)^{3/2} \exp\left[S\left(\rho^2 - \sum_{i=1}^A r_i^2\right)\right] \chi_{Ka}^*(\rho) \\ \times \sum_{i=1}^A \phi_\nu^*(\mathbf{r}_i) C^{*\nu r_i} (-1)^{i+\nu} \sum_{j=1}^A \delta(\mathbf{R} - \mathbf{r}_j) (-1)^{\nu+j} \phi_\nu(\mathbf{r}_j) C^{\nu r_j} \chi_{Ka}(\rho) \quad (29)$$

We now do the  $A-1$  integrals  $d\mathbf{r}_l$ , where  $l \neq j$  resulting in

$$d_\nu(R) = \frac{B^2}{A! \pi i} \int \frac{d\hat{R} d\rho ds}{\rho^{2K+3A-5}} \sum_{ij} \frac{\delta_{ij} (A-1)! e^{s(\rho^2 - r_i^2)} d\mathbf{r}_j \left(\frac{S}{\pi}\right)^{3/2}}{S^{3[(A-1)+K-l]/2}} \\ \times \chi_{Ka}(\rho) \phi_\nu^*(\mathbf{r}_j) \phi_\nu(\mathbf{r}_j) \chi_{Ka}(\rho) \quad (30)$$

The  $\delta_{ij} (A-1)!$  comes from integrating the two cofactors over all their coordinates.  $l$  is specified in equation (31) by  $\nu = (nlj m \tau z)$ . We assume  $n$  equals zero.

For our  $K$ min configuration of  $^{12}\text{C}$ ,  $l$  can be only zero or one. The contour integrals can be done as

$$\frac{1}{\pi i} \int_c \frac{ds}{s^M} e^{s(\rho^2 - r^2)} = \frac{2(\rho^2 - r^2)^{M-1}}{\Gamma(M)} \quad \text{if } \rho^2 > r^2 \quad (31)$$

$$= 0 \quad \text{if } r^2 > \rho^2 \quad (32)$$



Doing the sum on  $i$  and  $j$  results in

$$d_\nu(R) = \frac{B^2}{\pi^{3/2}\Gamma(M)} \int_R^\infty \frac{d\rho d\hat{R}}{\rho^{2K+3A-5}} \chi^2(\rho) \phi_\nu^2(r_1) \delta(\mathbf{R} - \mathbf{r}_1) 2(\rho^2 - r_1^2)^{M-1} d\mathbf{r}_1 \quad (33)$$

Here

$$M = K + \frac{3}{2}(A-1) - l - \frac{3}{2} \quad (34)$$

Doing the integrations  $d\mathbf{r}_1 d\hat{R}$  trivially, and the spin, isospin integrations indicated in equation (25), and the summations there over  $ml, ms$ , results in

$$d_\nu(R) = \frac{2B^2}{\pi^{3/2}\Gamma(M)} \int_R^\infty \frac{d\rho}{\rho^{2K+3A-5}} \chi^2(\rho) C_{nl}^2 R^{2(2n+l)} (\rho^2 - R^2)^{M-1} \quad (35)$$

Using equation (22) and equation (26), we can rewrite equation (35) as

$$d_\nu(R) = \frac{\Gamma(x) 2R^{2l}}{\Gamma(M)(l + \frac{1}{2})!} \int_R^\infty \frac{d\rho}{\rho^{2K+3A-5}} (\rho^2 - R^2)^{M-1} \chi^2(\rho) \quad (36)$$

These densities are normalized to

$$\int_0^\infty d_\nu(R) R^2 dR = \int_0^\infty \chi^2(\rho) d\rho = 1 \quad (37)$$

or

$$\int_0^\infty d(R) R^2 dR = A \quad (38)$$

The single-particle wave function can be deduced by setting

$$\varphi_\nu^2(R) = d_\nu(R) \quad (39)$$

We note if  $\chi^2(\rho) \approx \delta(\rho - a)$  then, from equation (36)

$$\varphi_\nu(R) \approx \frac{G}{a^{3/2}} \left(\frac{R}{a}\right)^l \left(1 - \frac{R^2}{a^2}\right)^{(M-1)/2} \quad (40)$$

If  $R \ll a$ , then

$$\varphi_\nu(R) \approx \frac{G}{a^{3/2}} \left( \frac{R}{a} \right)^l e^{-R^2(M-1)/2a^2}$$

Thus our single-particle wave functions resemble harmonic oscillator wave functions. This behavior does not continue indefinitely, however, because for large  $\rho$ ,  $\chi(\rho)$  the hyperradial dependence is exponential, i.e.,  $\chi(\rho) \propto e^{-\alpha\rho}$ , so that for large  $R$ ,

$$\varphi_\nu(R) \propto e^{-\alpha R} \quad (41)$$

Thus the single-particle wave functions have an exponential tail for large separations.

The spectral function for the entire nucleus can be written

$$S(W) = \int \Psi_a^* \delta(W - H) \Psi_a d\rho d\Omega = \delta(W - E_a) \quad (42)$$

if  $\Psi$  is an eigenfunction of  $H$ , with eigenvalue  $E_a$ , that is the solution of equation (16). The spectral function for a nucleon with quantum numbers  $\nu$  is

$$S_\nu(W) = S \Psi_a^* \delta(W - H_\nu) \Psi_a d\rho d\Omega \quad (43)$$

We now discuss the evaluation of equation (43).  $\Psi_a$  is not expected to be an eigenfunction of  $H_\nu$ . We note that  $H_\nu \Psi_a$  is a function of the coordinates  $(\rho, \Omega)$ . Thus we expand, using equation (15):

$$H_\nu \Psi_a = \sum_b C_{ba}^\nu R_a(\rho) Y_b(\Omega) \quad (44)$$

where

$$C_{ba}^\nu = \int d\Omega Y_b^*(\Omega) H_\nu Y_a(\Omega) R_a(\rho) / R_a(\rho) \quad (45)$$

We write equation (45) in this unusual fashion as  $H_\nu$  is a differential operator and can operate on the wave function  $R(\rho)$ . The expansion coefficients  $C_{ba}^\nu$  are functions of  $\rho$ , but not of  $\Omega$ . We utilize the fact that the eigenfunctions of a Hermitian operator (the Laplacian) form a complete set in the expansion, equation (44).

Using equation (45), we can cast the spectral energy function as

$$S_v(W) = \int R^*(\rho) Y_a(\Omega) \delta \left[ W - \sum_b C_{ba}^v Y_b(\Omega) / Y_a(\Omega) \right] \times R_a(\rho) Y_a(\Omega) d\rho d\Omega \rho^{3A-4} \quad (46)$$

To proceed further without approximation, one must solve

$$W \sum_b C_{ba}^v(\rho) Y_b(\Omega) / Y_a(\Omega) = 0 \quad (47)$$

for  $\rho$ . In general, the solution can be written as

$$\rho = \rho_e(W, \Omega) \quad (48)$$

The integral  $d\rho$  in equation (46) can now be done with the result

$$S_v(W) = \sum_e \int \frac{d\Omega Y_a^*(\Omega) \chi^2(\rho_e) Y_a(\Omega)}{|\{(\partial/\partial\rho)[\sum_b C_{ba}^v(\rho) Y_b(\Omega) / Y_a(\Omega)]\}_{\rho_e}|} \quad (49)$$

The sum over  $e$  in equation (49) is due to the possibility of there being several values of  $\rho$  which satisfy equation (47).

An approximate evaluation of equation (49) is now obtained as a series expansion in powers of  $C_{ba}/C_{aa}^v = x_b$ . The convergence of such an expansion will be discussed after we establish what the initial terms of the expansion are like.

Equation (47) is written as

$$W = C_{aa}^v(\rho_e) \left[ 1 + \sum_{b \neq a} X_b Y_b(\Omega) / Y_a(\Omega) \right] \quad (50)$$

To zeroth order in  $x_b$  the solution is independent of  $\Omega$ , and is

$$W = C_{aa}^v(\rho) \quad (51)$$

or

$$\rho = \rho_a(W) = C_{aa}^{-1}(W) \quad (52)$$

Comparing equations (50) and (52), we have

$$\rho_a = \rho_e \left[ 1 + \sum_{b \neq a} X_b Y_b(\Omega) / Y_a(\Omega) \right] \quad (53)$$

or

$$\rho_e \approx \rho_a - \rho_a \sum_{b \neq a} X_b Y_b(\Omega) / Y_a(\Omega) \quad (54)$$

From the integrand of equation (49), we now define

$$f(\rho_e) \equiv \frac{\chi^2(\rho_e)}{\{(\partial/\partial\rho)[C_{aa}^v(\rho) + \sum_{b \neq a} X_b Y_b(\Omega)/Y_a(\Omega)]\}_{\rho_e}} \quad (55)$$

and we do a Taylor series expansion on  $f$  as

$$f(\rho_e) = f(\rho_a) + \left. \frac{\partial f}{\partial \rho} \right|_{\rho_a} (\rho_e - \rho_a) + \mathcal{O}(x_b^2) \quad (56)$$

Substituting the expansion (56) into (49) we obtain

$$S_v(W) = \sum_e \int d\Omega Y_a^*(\Omega) Y_a(\Omega) \left[ f(\rho_a) + \left. \frac{\partial f}{\partial \rho} \right|_{\rho_a} (\rho_e - \rho_a) + \dots \right] \quad (57)$$

The term independent of  $xb$  integrates to

$$S_v(W) \approx \sum_e f(\rho_a) = \sum_e \frac{\chi^2(\rho_a)}{\{(\partial/\partial\rho)C_{aa}^v(\rho)\}_{\rho_a}} + \dots \quad (58)$$

The terms linear in  $xb$  integrate to zero as

$$\int Y_a^*(\Omega) Y_b(\Omega) d\Omega = \delta_{ab}$$

so that equation (58) is an approximation for the spectral energy function good to order  $x_b^2$ . We now discuss the evaluation of  $C_{aa}^v$  which is needed for the terms retained in the approximate spectral energy function evaluated, equation (58). We will subsequently show that  $x_b$  is small, so that the approximate formula, equation (58), can be expected to be rather accurate.

From equation (45), we have with  $b=a$

$$C_{aa}^v(\rho) = \int d\Omega Y_a^*(\Omega) H_v Y_a(\Omega) R_a(\rho) / R_a(\rho) \quad (59)$$

We separate  $H_\nu$  into  $T_\nu + V_\nu$ , kinetic and potential energy terms, where

$$C_{aa}^\nu(\rho) = \langle T_\nu \rangle + \langle V_\nu \rangle \quad (60)$$

Expanding the slater determinants  $P_a$  in cofactors we have

$$P_a = \sum_{i=1}^A \phi_{\nu_0}(r_i) C^{\nu_0 r_i} (-1)^{i+\nu_0} \quad (61)$$

Where  $C^{\nu_0 r_i}$  is an  $A-1$  by  $A-1$  determinant formed by deleting the  $i$ th coordinates and the  $\nu_0$ th wave function.

Using equations (20) and (21), and doing all the integrations  $d\mathbf{r}_i \neq d\mathbf{r}_i$  results in

$$\langle T_\nu \rangle = -\frac{\hbar^2 B^2}{2m\pi_i} \int \frac{ds d\mathbf{r} (s/\pi)^{3/2} e^{s(\rho^2-r^2)}}{\rho^{2K+3A-5} S^{K+3[(A-1)-l]/2}} \phi_\nu^*(\mathbf{r}) [\nabla_r^2 \phi_\nu(r) R_a(\rho)] / R_a(\rho) \quad (62)$$

Now  $\nabla^2 \phi R$  can be written as

$$\nabla^2 \phi R = \phi_\nu (A_0 + A_2 r^2) \quad (63)$$

where

$$A_0 = \frac{3}{\rho} \frac{\partial R}{\partial \rho} \quad (64)$$

and

$$\rho^2 A_2 = \frac{\partial^2 R}{\partial \rho^2} - \frac{1}{\rho} \frac{\partial R}{\partial \rho} \quad (65)$$

So that doing the integrals  $ds d\mathbf{r}$  and using equation (32), results in

$$\langle T_\nu \rangle = -\frac{\hbar^2 2 B^2 2 \{A_0 S_a + A_2 \rho^2 S_b\}}{2 M \pi^{3/2} (l + \frac{1}{2})! \Gamma(M) R_a} \quad (66)$$

where

$$M = K + \frac{3}{2}(A-2) - l \quad (67)$$

and where

$$S_a \equiv \sum_{s=0}^{M-1} \binom{M-1}{s} \frac{(-1)^s}{2l+2s+3} \quad (68)$$

$$S_b \equiv \sum_{s=0}^{M-1} \binom{M-1}{s} \frac{(-1)^s}{2l+2s+5} \quad (69)$$

We now evaluate the potential energy contribution to equation (61) as follows.

The potential term is written as

$$V = \frac{1}{2} \sum_{\substack{ij \\ \underline{\underline{=}}}} V(r_{ij}) \quad (70)$$

The wave function is expanded via equation (15) and  $Y_a(\Omega)$  is further expanded as in cofactors as

$$Y_a(\Omega) = \frac{B}{\rho^K (A!)^{1/2}} \sum_{i=1}^A \phi_{v_j}(\mathbf{r}_i) C^{v_j r_i} (-1)^{i+v_j} \quad \text{for any fixed } j \quad (71)$$

The cofactors are expanded in second cofactors as

$$C^{v_j r_i} = \sum_{\substack{j \neq j \\ \underline{\underline{=}}}}^A \phi_{v_j}(r_i) C C^{v_i v_j r_i} \quad \text{for any } \underline{\underline{i}} \neq \underline{\underline{i}} \quad (72)$$

Using equation (20) for  $d\Omega$  and integrating over all coordinates except those of  $d\mathbf{r}_1 d\mathbf{r}_2$ , one finally obtains

$$\langle V_v \rangle = D - E \quad (73)$$

Where

$$D \equiv \frac{B^2}{\pi i} \sum_{i=1}^A \int \frac{ds d\mathbf{r}_1 d\mathbf{r}_2}{\rho^{2K+3A-5}} \left( \frac{S}{\pi} \right)^{3/2} e^{s(\rho^2 - r_1^2 - r_2^2)} \phi_v^*(\mathbf{r}_1) \phi_{v_i}^*(\mathbf{r}_2) V(r_{12}) \phi_v(\mathbf{r}_1) \phi_{v_i}(\mathbf{r}_2) \quad (74)$$

and

$$E \equiv \frac{B^2}{\pi i} \sum_{i=1}^A \int \frac{ds d\mathbf{r}_1 d\mathbf{r}_2}{\rho^{2K+3A-5}} \left( \frac{S}{\pi} \right)^{3/2} e^{s(\rho^2 - r_1^2 - r_2^2)} \phi_v^*(\mathbf{r}_1) \phi_{v_i}^*(\mathbf{r}_2) V(r_{12}) \phi_v(\mathbf{r}_2) \phi_{v_i}(\mathbf{r}_1) \quad (75)$$

The integrals over  $ds$  in equations (74), (75) can be done analytically using equation (32) either before or after a Moshinsky-type transformation from  $\mathbf{r}_1, \mathbf{r}_2$  to relative and two-body center-of-mass coordinates is carried out. The final integrals over  $d\mathbf{r}_{12}$  can be done analytically for simple enough potentials forms, such as Yukawa, exponential or delta functions. Thus

$$C_{aa}^v = \langle T_v \rangle + \langle V_v \rangle \quad (76)$$

can be evaluated from an assumed potential,  $V(r_{ij})$ , and from a solution  $R_a(\rho) = \chi_a(\rho) / \rho^{(3A-4)/2}$  of the Schrödinger equation. The evaluation of  $C_{ba}$ , where

$$C_{ba} = \int d\Omega Y_b^*(\Omega) H_v Y_a(\Omega) R_a(\rho) / R_a(\rho) \quad (77)$$

analogous to equation (59), proceeds in a similar fashion. The Slater determinant in  $Y_b^*(\Omega)$  is expanded in cofactors. For the  $^{12}\text{C}$  ground state, we take the  $(S_{1/2})^4 (P_{3/2})^8$  configuration symbolized by the label  $a$ . We now discuss  $C_{ba}$  for various other configurations, labeled  $b$ .

We show that  $C_{ba}$  is small with respect to  $C_{aa}$  for various configurations  $b$ . For the ground state configuration assumed for  $^{12}\text{C}$ ,  $C_{ba} = 0$  unless the state  $b$  is a particle-hole excitation of state  $a$  where the particle is a  $(n+1 l j m \tau z)$  excitation and the hole is  $(n l j m \tau z) = \nu$ . More complicated particle-hole excitations vanish when hyperangularly averaged, and  $l j$  excitations fail to simultaneously conserve parity and angular momentum. We note  $n=0$  for the assumed ground state configuration  $a$ .

The kinetic energy contribution to  $C_{ba}$  vanishes as the state  $\nu$  and the particle-hole excitations differ.

Carrying through a similar analysis for  $C_{ba}$  we obtain

$$C_{ba} = \frac{B^2}{\pi i \rho^{2K+3A-5}} \int ds \left( \frac{s}{\pi} \right)^{3/2} e^{s(\rho^2 - r_1^2 - r_2^2)} d\mathbf{r}_1 d\mathbf{r}_2 \phi_\nu^*(r_1) \phi_{\nu_b}^*(r_2) V(r_{12}) \\ \times \{ \phi_\nu(r_1) \phi_{\nu_b}(r_2) - \phi_\nu(r_2) \phi_{\nu_b}(r_1) \} \quad (78)$$

Where  $\phi_\nu$  refers to the state associated with the knocked-out nucleon for which the spectral function is to be evaluated,  $\phi_\nu$  is an orbital in the assumed configuration for the ground state wave function,  $\phi_{\nu_b}$  is the  $n+1$  radial excitation of the orbital  $\phi_{\nu_a}$  and replaces  $\phi_{\nu_a}$  in configuration  $b$ .

To ensure configuration  $b$  is orthogonal to  $a$ , we write

$$Y_b(\Omega) = \frac{B}{\rho^K (A!)^{1/2}} \{ a_0 P^{n+1} + b_0 P^n \} \quad (79)$$

where  $P^n$  is the Slater determinant of configuration  $a$ , and  $P^{n+1}$  denotes  $n$  replaced by  $n+1$  for the state  $\nu$ . The coefficients guaranteeing orthonormality are

$$a_0 = -\frac{xb_0}{\rho^2} \left[ \frac{l+\frac{5}{2}}{l+\frac{3}{2}} \right]^{1/2} \quad (80)$$

and

$$b_0^2 = (x+1)\left(l+\frac{3}{2}\right) / \left(x-l-\frac{3}{2}\right) \quad (81)$$

where

$$x = K + \frac{3}{2}(A-1) \quad (82)$$

$l$  is the orbital angular momentum of the knocked-out nucleon.

To simplify the evaluation of  $C_{bc}/C_{aa}$ , we assume a delta function potential. Thus we assume

$$V(r_{12}) = -V_0 \delta^3(\mathbf{r}_1 - \mathbf{r}_2) \quad (83)$$

After considerable analysis of equations (74)–(79), we obtain

$$\frac{C_{ba}}{\langle V_\nu \rangle} \approx \frac{1}{A} \quad (84)$$

Now, for a delta function force  $\langle V_\nu \rangle$  is proportional to  $1/\rho^3$  so we cannot use a delta function force in calculations, but the ratio of  $C_{ba}/\langle V_\nu \rangle$  is independent of  $\rho$  for a delta function potential and is small.

For the more nearly realistic potentials used, the kinetic energy contribution is less than the potential energy contribution, as the values of energy for which the spectral energy function is nonvanishing, are all negative. Now

$$\frac{C_{ba}}{C_{aa}} = \frac{C_{ba}}{\langle T_\nu \rangle + \langle V_\nu \rangle} \quad (85)$$

We have calculated [cf. equation (84)]  $C_{ba}/\langle V_{\nu 0} \rangle$  as small, at least for a delta function potential.

$C_{aa}$  is approximately  $(3/4)\langle V_{\nu 0} \rangle$  for the potential parametrized to bind carbon 12 in Kmin approximation. Therefore, we infer that  $x_b = C_{ba}/C_{aa} \approx 1/A$  for the configuration of a radial excitation of a single



nucleon. Thus the expansion of the spectral energy function, equation (58), is probably good to order  $1/A$ , as the first-order terms all vanish and the corrections to the zeroth-order term is of second order in  $x_b$ . There are  $A$  of these correction terms, so the total correction is estimated as  $A(1/A)^2 = 1/A$ , compared to the zeroth-order term.

## RESULTS

Shown in Figure 2 is the predicted spectral energy function for  $l=0$  and for  $l=1$  nucleons in  $^{12}\text{C}$ . The spectral energy distributions are not sharp delta functions as assumed by the simple independent particle shell model. The  $l=0$  nucleons are bound from about 20 to 25 MeV, centered about 22 MeV binding energy. The energy is not discrete as the nucleons are interacting with other nucleons. Only for a two-body problem can one expect discrete bound energies. The  $l=0$  nucleons are bound at about 10.5 MeV and the distribution of  $p$  wave nucleons is much sharper than the  $l=0$  nucleons. It is speculated that this is due to the  $p$  wave nucleons interacting less than the  $S$  wave nucleons. Tending to be near the nuclear surface reduces the averaged two-body interaction for the  $l=1$  nucleons as they are further apart there. Thus the shell model assumption of sharp energy distributions for loosely bound nucleons has calculational support here. No other calculation able to lend direct support to that assumption is known to the author. Sharp peaks seen in reactions, i.e.,  $(p, d)$  and inelastic scattering, have been interpreted as lending support to the idea of discrete energy distributions for bound nucleons.

The question of how the nucleus can be a system of strongly interacting particles yet the independent particle model be valid has often been

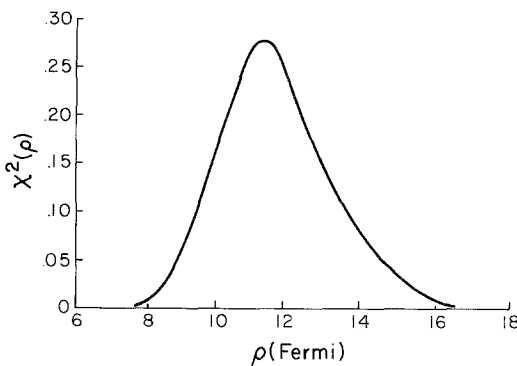


Fig. 2. The spectral energy function for  $l=0$  and 1 protons in the ground state of  $^{12}\text{C}$ .

puzzled over (De Shalit, 1963; De Shalit, 1974). We see an answer to that question developing here along the following lines. A collective strongly interacting system description of the nucleus was utilized to describe  $^{12}\text{C}$ . A two-nucleon potential is input to the problem. No one-body shell model potentials, nor effective residual potentials are utilized. No independent particle motion in an averaged potential is assumed. The center-of-mass motion is properly treated. Coming out of this calculation we see the  $l=0$  nucleons have a comparatively broad energy spread which we associate with strongly interacting many-body (more than 2) systems. The  $l=1$  nucleons have a narrow energy distribution due to an angular momentum barrier reducing their frequency of interaction.

We also speculate that the use of a spin-orbit term in the two-nucleon potential would increase the binding of the  $l=1$  nucleons relative to the  $l=0$  nucleons. This is desirable from reaction  $Q$  value systematics, which suggest  $P_{3/2}$  nucleons are bound by more than 10 MeV in  $^{12}\text{C}$ .

The single-particle wave functions can be deduced from the many-body wave function calculations and are similar to harmonic oscillator wave functions, except for an exponential tail at infinity, instead of a Gaussian behavior. The spectral function calculated depends on the target wave function used and the two-nucleon interaction assumed. The spectral energy function is calculated approximately, but not perturbatively. An expansion in a complete set of configurations is made and only the ground state configuration is retained. In this approximation the  $l=1$  protons have a sharply peaked spectral energy function. The  $l=0$  protons have a broader spectral energy function and are more tightly bound. While expected, it is very comforting to see this come out of a many-body calculation.

The widths of the calculated peaks are due to the use of a many-body theory and wave function, rather than the assumption of an independent particle shell model description.

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